REVERSE PROBLEM OF THE TRANSIENT GRADIENTAL FLOW OF A SHVEDOV–BINGHAM PLASTIC THROUGH A FLAT CHANNEL AND A CYLINDRICAL PIPE

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Analyzed is the reverse problem of the transient gradiental flow of a Shvedov–Bingham plastic through a flat channel and a cylindrical pipe. Equations are derived for determining the distribution of tangential shearing stresses and for the pressure gradient as a function of time, these equations to be solved by an iteration scheme which is shown here.

Viscoplastic media represent one of the most important rheological models of a continuous medium in petroleum mechanics. The state of the art in research on the hydrodynamics of viscoplastic media (Shvedov–Bingham plastics) has been surveyed in [1, 2]. The problem of the transient gradiental flow of a Shvedov–Bingham plastic has been solved in [3] by the method of successive approximations. Here we will use the method of successive approximation for solving the reverse problem of the transient gradiental flow of a viscoplastic medium.

We consider the following problem. A viscoplastic medium with the density ρ, the yield shearing stress τ_0, and the dynamic viscosity μ (rheological parameters of the medium) begins at time $t < 0$ to flow through a flat channel ($-a < x < a$) or a cylindrical pipe ($0 < x < a$) at a velocity $u = u(x, t)$ (x denoting the transverse coordinate) due to a pressure gradient $\varphi(t)$ which varies in time. The flow equation for a continuous medium will be

$$\rho \frac{\partial u}{\partial t} = + \varphi(t) + \frac{1}{x^k} \frac{\partial}{\partial x} x^k \tau, \tag{1}$$

with $\tau = \tau(x, t)$ denoting the distribution of tangential shearing stresses and with $k = 0$ for the plane case or $k = 1$ for the cylindrical case (axial symmetry). The rheological law for a viscoplastic medium in one-dimensional rectilinear motion with $\partial u/\partial x < 0$ (in the plane case we consider the upper half of the channel) becomes

$$\tau = \mu \frac{\partial u}{\partial x} - \tau_0, \quad |\tau| \geqslant \tau_0, \tag{2}$$

$$\frac{\partial u}{\partial x} = 0, \quad |\tau| < \tau_0. \tag{3}$$

The channel wall at $x = a$ remains stationary throughout the time interval under consideration and, therefore, according to the hypothesis that the fluid adheres to the wall, we have

$$u(a, t) = 0. \tag{4}$$

as the kinematic constraint on the velocity of the fluid medium. Integrating Eq. (2) with respect to variable x and considering the condition at the channel wall, we have

$$u(x, t) = \frac{1}{\mu} \int_a^x \tau \, dx + \frac{\tau_0}{\mu}(x - a). \tag{5}$$

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Relation (5) will yield the velocity of the fluid, if the distribution of tangential shearing stresses is known.

The flow equation (1) can be rewritten as

$$\frac{\rho}{\mu}\int_a^x \frac{\partial\tau}{\partial t}\,dx = +\varphi(t) + \frac{1}{x^k}\frac{\partial}{\partial x}(x^k\tau). \tag{6}$$

The condition for the existence of a quasirigid core in the stream within $x \le \delta(t)$ is the first condition at the sought boundary:

$$\tau(x,\,t)|_{x=\delta(t)} = -\tau_0. \tag{7}$$

This quasirigid core in the stream moves as a single body and, as a consequence, we have

$$\frac{1}{x^k}\frac{\partial}{\partial x}(x^k\tau)\bigg|_{x=\delta(t)} = -(k+1)\frac{\tau_0}{\delta(t)}. \tag{8}$$

Since the flow evolved from a condition of rest, when the entire region had been quasirigid, hence the following initial condition applies to $\delta(t)$:

$$\delta(0) = a. \tag{9}$$

We will now consider more thoroughly the second condition (8) at the sought boundary. Motion in the quasirigid zone is analogous to the motion of a perfectly rigid body, and (1) with (3) will yield

$$\frac{1}{x^k}\frac{\partial}{\partial x}(x^k\tau) = f(t), \tag{10}$$

where the certain function f(t) is yet to be determined. The general solution to Eq. (10) is

$$\tau = f(t)\frac{x}{k+1} + x^{-k}c(t), \tag{11}$$

with c(t) denoting an arbitrary function. The tangential shearing stress vanishes at $x \to 0$ and, therefore, $c(t) \equiv 0$. Considering condition (7) at the sought boundary, we obtain the distribution of tangential shearing stresses within the quasirigid flow zone:

$$\tau = -\tau_0\frac{x}{\delta(t)}. \tag{12}$$

Relation (8) can be easily derived from (12), if one considers that the velocity of the fluid and the distribution of tangential shearing stresses change smoothly at the boundary between both flow zones $x = \delta(t)$.

We now introduce the dimensionless quantities

$$\bar{x} = \frac{x}{a};\ \Delta = \frac{\delta}{a};\ \tau = \frac{\tau}{\tau_*};\ s = \frac{\tau_0}{\tau_*};$$
$$\bar{t} = \frac{\mu}{\rho a^2}t;\ \bar{\varphi} = \frac{\varphi\cdot a}{\tau_*},$$

where τ_* is a characteristic stress in this problem, equal to $a\varphi_\infty$. In dimensionless form, the problem can be formulated as follows (the dashes over symbols for dimensionless quantities will be omitted):

$$\frac{1}{x^k}\frac{\partial}{\partial x}(x^k\tau) = -\varphi(t) + \int_1^x \frac{\partial\tau}{\partial t}dx, \tag{13}$$

$$\tau(x,\,t)|_{x=\Delta(t)} = -s, \tag{14}$$

$$\frac{1}{x^k}\frac{\partial}{\partial x}(x^k\tau)\bigg|_{x=\Delta(t)} = -(k+1)\frac{s}{\Delta(t)}, \tag{15}$$

$$\Delta(0) = 1. \tag{16}$$

Let us then consider the so-called reverse problem: assuming the function $\Delta = \Delta(t)$ to be known, we are to determine the distribution of tangential shearing stresses within the viscous flow zone $\Delta(t) < x < 1$

and to establish how the pressure gradient must vary with time so that such a flow will result.

From (13) at $x = \Delta(t)$, with condition (15) at the sought boundary, we obtain

$$\varphi(t) = (k+1)\frac{s}{\Delta(t)} + \int_1^{\Delta} \frac{\partial \tau}{\partial t}\, dx. \tag{17}$$

Noting that $\int_1^x \frac{\partial \tau}{\partial t} dx = \int_1^{\Delta} \frac{\partial \tau}{\partial t} dx + \int_{\Delta}^x \frac{\partial \tau}{\partial t} dx$ and using relation (17), we obtain

$$\frac{1}{x^k}\frac{\partial}{\partial x}(x^k \tau) = -(k+1)\frac{s}{\Delta(t)} + \int_{\Delta}^{x} \frac{\partial \tau}{\partial t}\, dx.$$

Multiplying both sides of this equation by x^k and then integrating the result with respect to x from Δ to x, with condition (14) taken into account, we obtain, after simple calculations,

$$\tau = -s\frac{\Delta^k}{x^k} - \frac{s}{\Delta}\frac{x^{k+1} - \Delta^{k+1}}{x^k} + \frac{1}{x^k}\int_{\Delta}^{x} x^k \int_{\Delta}^{x} \frac{\partial \tau}{\partial t}\, dx dx. \tag{18}$$

The integrodifferential equation (18) can, for sufficiently smooth functions $\Delta = \Delta(t)$, be solved by the method of successive approximation with the following scheme of iterations:

$$\tau^{(n)} = -s\frac{\Delta^k}{x^k} - \frac{s}{\Delta}\frac{x^{k+1} - \Delta^{k+1}}{x^k} + \frac{1}{x^k}\int_{\Delta}^{x} x^k \int_{\Delta}^{x} \frac{\partial \tau^{(n-1)}}{\partial t}\, dx dx,$$
$$n = 1,\ 2,\ 3\ \ldots, \tag{19}$$
$$\tau^{(0)} = -s\frac{\Delta^k}{x^k} - \frac{s}{\Delta}\frac{x^{k+1} - \Delta^{k+1}}{x^k}.$$

Inserting the calculated values of $\tau^{(n)}$ into (17), we obtain the sequence of functions $\varphi^{(n)}(t)$:

$$\varphi^{(n+1)}(t) = (k+1)\frac{s}{\Delta(t)} + \int_1^{\Delta} \frac{\partial \tau^{(n)}}{\partial t}\, dx,$$
$$\varphi^{(0)} = (k+1)\frac{s}{\Delta(t)}. \tag{20}$$

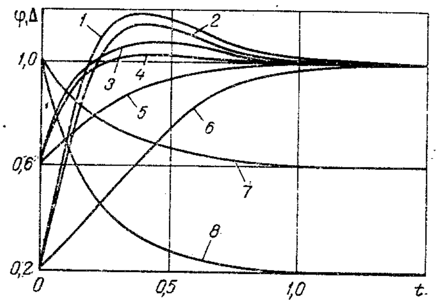

Fig. 1

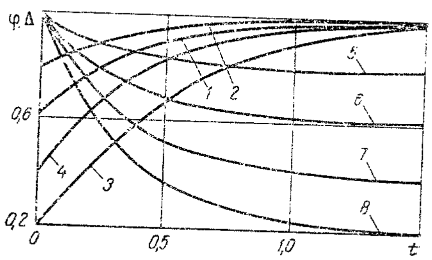

Fig. 2

Fig. 1. Qualitative convergence pattern of the iteration process, with $\alpha = 5$: 1) φ_1 (s = 0.2); 2) φ_2 (s = 0.2); 3) φ_1 (s = 0.6); 4) φ_2 (s = 0.6); 5) φ_0 (s = 0.6); 6) φ_0 (s = 0.2); 7) Δ (s = 0.2); 8) Δ (s = 0.6).

Fig. 2. Characteristics of plane flow, with $\alpha = 3$: 1–4) φ_2; s = 0.8; 0.6; 0.4; 0.2, respectively; 5) s = 0.8; 6) 0.6; 7) 0.4; 8) 0.2.

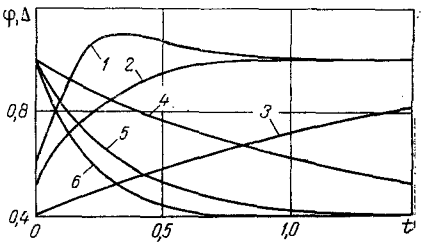

Fig. 3

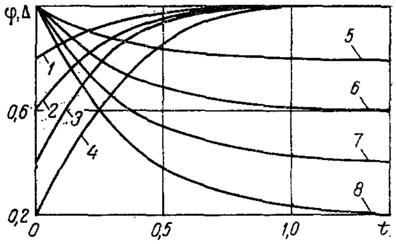

Fig. 4

Fig. 3. Effect of parameter α on the flow characteristics in a plane channel, with s = 0.4: 1-3) φ_2, α = 5; 3; 1, respectively; 4-6) Δ, α = 1; 3; 5.

Fig. 4. Characteristics of cylindrical flow, with α = 3: 1-4) φ_2, s = = 0.3; 0.4; 0.1; 0.2, respectively; 5-8) Δ, s = 0.4; 0.3; 0.2; 0.1.

For specific results in terms of the displacement of the boundary between viscous and quasirigid flow as a function of time, we use the relation

$$\Delta = (1+k)s + [1-(1+k)s]\,e^{-\alpha t}. \tag{21}$$

This function approaches unity as the limit at $t \to 0$, according to condition (16), and represents the transition to steady-state flow at $t \to \infty$. The maximum value

$$\Delta_s = (1+k)s \tag{22}$$

corresponds to steady flow of a viscoplastic medium due to a constant pressure gradient. Parameter α characterizes the speed at which steady state is reached. The convergence of the iteration process is depicted in Fig. 1 on a qualitative basis. In Fig. 2 are shown values based on the third approximation for $\varphi = \varphi(t)$ in the plane case with $\alpha = 3$, in Fig. 3 is shown the effect of parameter α on the variation of the pressure gradient with time, and in Fig. 4 are shown $\varphi(t)$ curves for the cylindrical case.

NOTATION

t is the time;
ρ is the density;
τ_0 is the yield shearing stress;
μ is the dynamic viscosity;
u is the velocity;
x is the space coordinate;
a is the characteristic channel dimension;
φ is the modulus of the pressure gradient;
k is the symmetry parameter in the problem;
τ is the tangential shearing stress;
δ, Δ are the boundary between viscous flow and quasirigid flow zone;
f, c are the arbitrary functions;
τ_* is the characteristic stress in the problem;
s is the plasticity parameter;
n is the order number of iteration;
α is the time constant of the test function;
Δ_s is the maximum value of $\Delta(t)$.

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